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Likelihood Analysis of Random Effect Stochastic Frontier Models with Panel Data

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Abstract

The paper takes up posterior analysis of the stochastic frontier model with random effects when panel data is available. Available treatments of the model result in a likelihood function that is highly nonlinear and, as a result, applied researchers prefer to use fixed effect formulations when efficiency measurement is sought from panel data. The methodology is based on Gibbs sampling. It is shown how posterior distributions of parameters can be derived and how firm-specific efficiency measures can be computed.

1. Introduction

Starting with the pioneering work of Aigner, Lovell and Schmidt (1977), the use of stochastic frontier models has a long-standing tradition in econometrics and such models have been used with success in numerous applications. See Bauer (1990) and Greene (1993) for a survey of the literature. Well known assumptions about the distribution of the one-sided disturbance term in stochastic frontier models, include the exponential of Aigner et al. (1977), the exponential of Meeusen and van den Broeck (1977) and the truncated normal of Stevenson (1980) which includes the half-normal as a special case.

When panel data is available, a number of possibilities open for the estimation of stochastic frontier models, including fixed effect or random effect estimation. The major advantage of having panel data is that estimated firm-specific efficiencies are consistent which is not the case in cross-section models. The typical approach to frontier estimation with panel data is to use fixed effect estimation and then estimate inefficiency by taking the difference of each firm-specific coefficient from its maximum value (Greene, 1980). Pitt and Lee (1981) and Battese and Coelli (1988) have taken up maximum likelihood estimation of the stochastic frontier model with random effects.

Possibly because of the complexity of the likelihood function, applied researchers tend to rely on fixed effect formulations. The purpose of this paper is to present posterior analysis of the random effect frontier model organized around Markov Chain Monte Carlo simulation techniques, especially the Gibbs sampler. Posterior distributions of parameters and efficiency measures can be derived routinely using this technique.

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2. The Model

Consider the stochastic frontier model for a (possibly unbalanced) panel data set

$$Y_{it} = \chi'_{it}\beta + v_{it} - u_i \tag{1}$$

$$t = 1,..,T_i,\ \ i = 1,..,N$$

where y_{it} is typically the log of output, x_{it} is a $k \times 1$ vector of explanatory variables, β is a $k \times 1$ vector of parameters, v_{it} is a two-sided error term representing random noise and u_i is a positive random variable representing technical inefficiency. In order to exploit the panel structure of the data, technical inefficiency is assumed time invariant. It is possible, however, to add time effects in order to separate technical inefficiency from factors, which are time varying. Regarding the error terms the following assumptions are introduced.

1. v_{it} is $N(0,\ \sigma_v^2)$

2. u_i is $N(0,\ \sigma_u^2)$

3. v_{it} and u_i are independent as well as independent of x_{it}.

The likelihood function of the model has been presented by Pitt and Lee (1981) and Battese and Coelli (1988) as

$$L(\beta, \lambda, \sigma_v; y, X) = \sum_{i=1}^{N}\{-\tfrac{1}{2}[T_i ln 2\pi - ln 2 + T_i ln\, \sigma_v^2 + ln(1 + \lambda T_i)]\}$$

$$+\sum_{i=1}^{N}(-\tfrac{1}{2}\{\frac{-\lambda}{1+\lambda T_i)}[\sum_{t=1}^{T_i}\frac{\varepsilon_{it}}{\sigma_v}]^2 + \sum_{t=1}^{T_i}[\frac{\varepsilon_{it}}{\sigma_v}]^2\}) \tag{2}$$

$$+\sum_{i=1}^{N} ln\,\Phi\{(\frac{\lambda}{1+\lambda T_i})^{1/2}\sigma_v^{-1}\sum_{t=1}^{T_i}\varepsilon_{it}\}$$

where $\lambda = \sigma_u^2/\sigma_v^2$ and $\varepsilon_{it} = y_{it} - x'_{it}\beta$. To make use of a Gibbs sampling approach to posterior inference we notice that the joint distribution of y_{it} and u_i is given by

$$f(y_{it}, u_i \mid \beta, \sigma_u, \sigma_v, y, X) = f_N^1(y_{it} \mid x'_{it} - u_i,\ \sigma_v^2)\, f_N(u_i \mid 0,\ \sigma_u^2)\ 1(ui \geq 0) \tag{3}$$

where $f_N^k(x \mid \mu, \Sigma)$ denotes the $k-$variate normal distribution with mean μ and covariance matrix Σ, and $1(A)$ denotes the indicator function of event A. If we treat the u_i's as parameters whose prior is a truncated normal the augmented likelihood function of the model becomes

$$L(\beta,\sigma_u,\sigma_v;\mathrm{y},\mathrm{X})=\prod_{i=1}^{N} f_N^{T_i}(\mathrm{y}_i \mid \mathrm{X}_i\beta-u_i 1_{T_i},\sigma_v^2) f_N(u_i \mid 0,\sigma_u^2)=$$
$$\sigma_v^{-(NT+1)} exp\{-\frac{(\mathrm{y}-\mathrm{X}\beta+\mathrm{U})'(\mathrm{y}-\mathrm{X}\beta+\mathrm{U})}{2\sigma^2}\}\sigma_u^{-(N+1)} exp(-\frac{\mathrm{u'u}}{2\sigma^2}) \tag{4}$$

where $y_i=[y_{i1}\ y_{i2}\ ...\ y_{iT_i}]'$ is $T_i\times 1$, $\mathrm{X}_i=[\mathrm{x}_{i1}\ \mathrm{x}_{i2}\ ...\ \mathrm{x}_{iT_i}]'$ is $T_i\times k$, $\mathrm{y}=[\mathrm{y}_1\ \mathrm{y}_2\ ...\ \mathrm{y}_N]'$ is $NT\times k$ where $T=\sum_{i=1}^{N}T_i$, $\mathrm{X}=[\mathrm{X}_1'\ \mathrm{X}_2'\ ...\ \mathrm{X}_N']'$ is $NT\times k$, $\mathrm{u}=[\mathrm{u}_1\ \mathrm{u}_2\ ...\ \mathrm{u}_N]'$ is the $N\times 1$ vector of inefficiencies, and $\mathrm{U}=\mathrm{e}\otimes\mathrm{u}$, with $\mathrm{e}=[1_{T_1}'\ 1_{T_2}'\ ...\ 1_{T_N}']'$, and 1_n denotes the unit vector in $\Re^n$.

To proceed with posterior analysis, a prior is chosen that is flat for β but inverted gamma for the variance parameters.

$$p(\beta,\sigma_u,\sigma_v)\propto\sigma_v^{-(N_1+1)} exp(-\frac{q_1}{2\sigma_v^2})\sigma_u^{-(N_2+1)} exp(-\frac{q_2}{2\sigma_u^2}) \tag{5}$$

The posterior distribution may be analyzed using the Gibbs sampler. The Gibbs sampler (Gelfand and Smith, 1989, Tanner and Wong, 1987 and Tierney, 1994) is an iterative scheme that generates a set of parameter draws $\{\beta^{(i)},\sigma_u^{(i)},\sigma_v^{(i)};i=1,..,M\}$ which converge in distribution to the posterior. These parameter draws are generated by drawing successively from the posterior conditional distribution of $\beta\mid\sigma_v,\sigma_u,u,y,X$, the posterior conditional distribution of $\sigma_v\mid\beta,\sigma_u,u,y,X$ and so forth. For a posterior distribution $\pi(\theta \mid \mathrm{D})$ where D denotes the data and θ denotes the entire parameter vector, the Gibbs sampler starts from a given initial parameter vector $\theta(0)$ and produces a set of parameter draws $\{\theta(i)\ \ i=1,...,M\}$ that converge in distribution to $\pi(\theta \mid \mathrm{D})$ under fairly weak conditions (Roberts and Smith, 1994). These random drawings are produced as follows. For i=1,...,M:

Draw $\theta_1^{(i)}$ from $\pi(\theta_1 \mid \theta_{-1}^{(i-1)},\mathbf{D})$,

draw $\theta_2^{(i)}$ from $\pi(\theta_2 \mid \theta_{-2}^{(i-1)},\mathbf{D})$,

(……..)

draw $\theta_k^{(i)}$ from $\pi(\theta_k \mid \theta_{-k}^{(i-1)},\mathbf{D})$

where $\theta_{-i}=[\theta_1 \ldots \theta_{i-1}\ \theta_{i+1} \ldots \theta_k]$.

The Bayesian approach to frontier models is presented in van den Broeck et al (1994) and Gibbs sampling in frontier models has been initiated with Koop et al (1995), see also the review paper by Osiewalski and Steel (1998). In this model, these posterior conditional distributions have standard forms from which random number generation is particularly easy. The procedure is described next.

3. Conditional Distributions for Gibbs Sampling

The conditional distribution of β is given by

$$\beta \mid \sigma_u, \sigma_v, \mathrm{u}, \mathrm{y}, \mathrm{X} \sim N_k((\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'(\mathrm{y}+\mathrm{U}), \sigma_v^2(\mathrm{X}'\mathrm{X})^{-1}) \tag{6}$$

The conditional distributions of the variance parameters may be reduced to χ^2 as usual and are given by the following.

$$\frac{\underline{q}_1 + (\mathrm{y}+\mathrm{U}-\mathrm{X}\beta)'(\mathrm{y}+\mathrm{U}-\mathrm{X}\beta)}{\sigma_v^2} \sim \chi^2(NT+\underline{N}_1) \tag{7}$$

and

$$\frac{\underline{q}_2 + \mathrm{u}'\mathrm{u}}{\sigma_u^2} \sim \chi^2(N+\underline{N}_2) \tag{8}$$

from which random number generation is straightforward. Contrary to the methods of Pitt and Lee or Battese and Coelli (1988) no reparametrization is needed in order to obtain positive estimates of the variance parameters. Drawing these parameters as above automatically guarantees that they are positive.

Finally, the conditional posterior distribution of u_i is a truncated normal. To see this, notice that because of the independence of v_{it}'s across i this posterior conditional can be expressed as

$$p(u_i \mid \beta, \sigma_u, \sigma_v, \mathrm{y}, \mathrm{X}) \propto exp\{-\frac{(\mathrm{y}_i - \mathrm{X}_i\beta + u_i 1_{T_i})'(\mathrm{y}_i - \mathrm{X}_i\beta + u_i 1_{T_i})}{2\sigma_v^2} - \frac{u_i^2}{2\sigma_u^2}\}\ 1(u_i \geq 0) \tag{9}$$

This apparently reduces to a normal distribution truncated to the positive half line. Its moments are given by

$$E(u_i \mid \beta, \sigma_u, \sigma_v, \mathrm{y}, \mathrm{X}) = \frac{T_i \sigma_u^2}{T_i \sigma_u^2 + \sigma_v^2} h_i \tag{10}$$

where

$$h_i = T^{-1} \sum_{t=1}^{T_i} (\mathrm{x}'_{it}\beta - y_{it}) \tag{11}$$

and

$$Var(u_i \mid \beta, \sigma_u, \sigma_v, \mathrm{y}, \mathrm{X}) = \frac{\sigma_v^2 \sigma_u^2}{T_i \sigma_u^2 + \sigma_v^2} \tag{12}$$

Drawings from this truncated normal distribution can be realized by using a special acceptance method described in Tsionas (2000).

4. Marginal Posterior Distributions and Efficiency Measurement

For a general kernel posterior distribution $p(\theta \mid \mathrm{D})$ where $\theta \in \Theta \subseteq \Re^k$ is a $k-$ dimensional parameter vector, marginal posterior distributions of any subset of the parameters may be derived routinely by utilizing the Gibbs sampling draws. If the parameter vector can be partitioned as $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ the marginal posterior distribution of the first component is given by

$$p(\theta_1 \mid \mathrm{D}) \propto \int p(\theta) d\theta_2 \tag{13}$$

which may be approximated by

$$p(\theta_1 \mid \mathrm{D}) \propto M^{-1} \sum_{i=1}^{M} p(\theta_1, \theta_2^{(i)}) \tag{14}$$

which is computed pointwise in θ_1. The integration constant which makes the above an exact density, may be computed using numerical integration, particularly a Simpson rule which requires only that the function is available over a set of ordinates.

To compute firm-specific efficiency measures, we can use (9), which gives the conditional distribution of u_i given the parameters and the data. The conditioning with respect to the parameters is not helpful, and this uncertainty must be eliminated in standard Bayesian fashion by integrating the parameters out of this conditional distribution. Therefore we obtain the distribution

$$\begin{aligned} p(u_i \mid \mathrm{y}, \mathrm{X}) &= \int p(u_i \mid \beta, \sigma_u, \sigma_v, \mathrm{y}, \mathrm{X}) d\beta d\sigma_u d\sigma_v \propto \\ &\int exp\{-\frac{(\mathrm{y}_i - \mathrm{X}_i\beta + u_i 1_{T_i})'(\mathrm{y}_i - \mathrm{X}_i\beta + u_i 1_{T_i})}{2\sigma_v^2} - \frac{u_i^2}{2\sigma_u^2}\}\, 1(u_i \geq 0) d\beta d\sigma_u d\sigma_v \end{aligned} \tag{15}$$

This gives the posterior distribution of the inefficiency component conditionally only on the observed data. Efficiency may be defined as the random variable

$$r_i = exp(-u_i), \;\; 0 \leq r_i \leq 1 \tag{16}$$

The distribution of the efficiency measure conditionally on the observed data may be derived from (15) by using a change of variables, namely

$$p(r_i \mid \mathrm{y},\mathrm{X}) = p_u(-log\, r_i \mid \mathrm{y},\mathrm{X}) r_i^{-1}, \quad 0 \le r_i \le 1 \tag{17}$$

A useful efficiency measure is the expected value of this distribution, which may be computed as

$$E(r_i \mid \mathrm{y},\mathrm{X}) = \int_0^1 p(r_i \mid \mathrm{y},\mathrm{X}) dr_i = \int_0^1 p_u(-log\, r_i \mid y_i, \mathrm{x}_i) dr_i \tag{18}$$

To evaluate (15) we apply the technique in (14) and average over the Gibbs draws to obtain the following approximation:

$$p(u_i \mid \mathrm{y},\mathrm{X}) =$$

$$M^{-1} \sum_{j=1}^{M} exp\{-\frac{(\mathrm{y}_i - \mathrm{X}_i\beta^{(j)} + u_i^{(j)} 1_{T_i})'(\mathrm{y}_i - \mathrm{X}_i\beta^{(j)} + u_i^{(j)} 1_{T_i})}{2\sigma_v^{2(j)}} - \frac{u_i^{2(j)}}{2\sigma_u^{2(j)}}\}\ 1(u_i^{(j)} \ge 0) \tag{19}$$

where the superscript (j) indicates that the parameter is evaluated at the j th Gibbs draw.

Alternatively, we may notice that (10) provides the conditional expectation of inefficiency component given the parameters and data. Therefore, we can apply the technique in (14) to obtain average inefficiency as a sum of (10) with respect to the parameter draws

Conclusions

The present paper derived the posterior distribution of parameters in a stochastic frontier model with random effects, when panel data is available. Although the posterior distribution is highly nonlinear, specialized methods can be used to obtain random draws that converge in distribution to the posterior. These methods are organized around Gibbs sampling with data augmentation. The method can be used to obtain exact, finite sample posterior distributions of parameters, as well as firm-specific efficiency mesures.

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