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Product Price Model Based on Random Exponential Pattern of Consumer Population Behaviour

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Abstract:

Purpose: *The main purpose of the article is to find the dynamic stability of the market equilibrium point under the specific assumptions of the consumer's surplus. The provided analysis takes advantage of the supply and demand for a hypothetical good using statistical methods.*
Design/Methodology/Approach: *The research is based on the mathematical model with statistical distributions of selected microeconomic random variables. The equations refer to the consumer's willingness to buy a good (a deterministic equation), the purchase probability (a stochastic equation) and the price of the good as a function of the marginal cost and the fraction of consumers who buy the good (a deterministic equation). The three above equations are interdependent. The price equilibrium point is found using numerical approximations of the final equation.*
Findings: *The model estimates the equilibrium price for a hypothetical good. In this paper it is assumed that the consumer may buy a product even if his or her surplus is negative. Moreover, a dynamic component, i.e. the derivative of the price over time has been added.*
Practical implications: *The dynamic stability of the equilibrium point has been mathematically proven. The paper also contains the application of the model and the equilibrium price to the case of the Polish mortgage market in the fourth quarter of 2022.*
Originality value: *The proposed rule is general and can be applied to other goods and markets.*

Keywords: *Demand and supply curves, willingness to pay, consumer surplus, exponential distribution, dynamic stability.*

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1. Introduction

The main purpose of the article is to find the dynamic stability of the market equilibrium point under the specific assumptions of the consumer's surplus. The provided analysis takes advantage of the supply and demand for a hypothetical good using statistical methods. Although the supply and demand curves are not explicitly estimated, they are derived from the marginal cost and the probability of the good purchase.

The topic is still important in the microeconomic research as empirical studies regarding the demand and supply curves as scarce compared to the existing theory. It creates the room for economic research. In this paper it is assumed that the consumer may buy a product even if his or her surplus is negative. Moreover, a dynamic component, i.e., the derivative of the price over time has been added. Those two represent a new approach in the microeconomic research and are the most important value added of the paper. Theoretical results have been tested in the context of the Polish mortgage market.

The research is based on the mathematical model with statistical distributions of selected microeconomic random variables. The model estimates the equilibrium price for a hypothetical good. The equations refer to the consumer's willingness to buy a good (a deterministic equation), the purchase probability (a stochastic equation) and the price of the good as a function of the marginal cost and the fraction of consumers who buy the good (a deterministic equation).

The three above equations are interdependent. The price equilibrium point is found using numerical approximations of the final equation. Moreover, the dynamic stability of the equilibrium point has been mathematically proven.

The article includes the following sections: Literature review, Research methods, Results, Discussion and Conclusions. The Literature review section presents the works starting from the neoclassical economics until current papers on demand or supply estimation including the willingness-to-pay approach.

The research methods section describes the assumptions of the three models and suggests a possible way of solving the problem. It also contains a comprehensive list of all notations and assumptions. In the theoretical results section the respective theorems that describe the properties of the product purchase probability and equilibrium price are included.

All the propositions are accompanied with their proofs and finally the equilibrium point is found numerically. The Empirical results section contains the application of the model and the equilibrium price to the case of the Polish mortgage market in the fourth quarter of 2022. In the Discussion section some possible extensions of the model are suggested. The Conclusion section summarizes all theoretical and empirical results.

2. Literature Review

Alfred Marshall introduced the idea that the demand, supply, costs and price are interdependent (Marshall, 2013). Using his *ceteris paribus* clause, he contributed greatly to the economic theory by creating the diagrams of demand and supply. Although the microeconomic equilibrium where the demand line crosses the supply line is static, the dynamic stability can also be analysed using that idea.

According to Marshall the concept of "invisible hand" can be applied in the partial equilibrium theory as the excess demand makes the price automatically increase and the excess supply makes its price decline. Nevertheless, the exact estimation of the demand and supply curves has proved to be quite challenging ever since. Marshall also widely used the idea of the consumer's surplus (or rent), i.e., the difference between the maximum price that the consumer would be willing to pay for the good and the price that he or she actually pays.

However, he took advantage of previous works by Cournot (1927) and Dupuit (1969). Moreover Jenkin (1872) is believed to have introduced a similar idea independently (Dooley, 1983).

Moore (1919) discovered the relationship between the percentage change in the price and the percentage change in the amount of production, and between the ratio of the production of a given year to the production of the previous year and the ratio of the price of the given year to the price of the preceding year. It was assumed that both relationships are polynomials of degree 3. The law was illustrated with data from cotton production.

Working (1925) described different data and statistical approaches to find the demand curve as a function of the price. Another important paper related to the relationship between demand, supply and price was the paper published by Gale in 1955, where the author analysed the properties of the relationship between the supply and demand. In particular, he proved the equilibrium theorem and some conditions that guarantee the uniqueness and optimality of the equilibrium price vector.

The topic of the moving equilibrium of demand and supply was raised by Moore (1925). He analysed two different types of the supply and demand curves whose parameters were estimated by the least square method. The approaches were illustrated with data on the annual production of potatoes and their farm prices in USA. Whelan and Msefer (1996) went beyond the static approach that had been prevalent in the analysis of supply and demand.

The paper introduced a model that incorporated elements from the classical economics as well as several real-world assumptions. This model is further used to examine some of the interactions among supply, demand and price. To explore the dynamics of supply and demand the authors used the clothing market as an example. Such classical approaches do not explain how the equilibrium point is reached or how to find the time dependence between the price and the equilibrium point.

In the dynamic models of the microeconomic equilibrium it was also assumed that the time is discrete rather than continuous. In the Arrow-Hurwicz (1958) model the global supply of goods was compared with the global demand. If the former is greater than the latter, the prices of respective goods decline. In the opposite situation the prices increase. An interesting example of that approach is the so-called cobweb model (Nerlove 1958), where the actual price oscillates around the equilibrium price. But unlike the neoclassical approach, the oscillations need not be damped.

They can be uniform or even explosive, depending on the adjustment path between consecutive time periods. That model may be easily supplemented with price floors or price ceilings that partially modify the oscillation path. In many microeconomic models the demand is considered to be deterministic and, for simplicity, also linear (Bolton and Dewatripont, 2005). In that case the actual price can be set somewhere in between the marginal cost and the consumer's willingness to pay. A soft constraint is possible here.

However, the demand may also contain the probabilistic part (McFadden, 1973) which can be a consequence of e.g. random utility or an error (Chakravarty, 1999; Kumar *et al.,* 2009). An interesting extension of the willingness-to-pay approach represents the assumption that a particular consumer may consider its reservation price for a product as a range, rather than a precise value (Wang *et al.,* 2007). It may stem from the fact that the consumer is actually uncertain about his or her preferences.

Dost and Wilken (2012) show that it might be better to use a range-based approach compared to a point-based approach. However, uncertain consumers are not the same group as indecisive consumers (Dost *et al.,* 2014). There are also other applications of this approach to the case of bucket pricing (Schlereth and Skiera, 2012) or multi-product distributions of consumers' price responses (Dost and Geiger, 2017). Changes in consumer metrics influence both the customer equity (i.e., the sum of all customers' lifetime values) and the shareholder value (Schulze *et al.,* 2012). Finally, Liozu *et al.* (2012) are in favour of more research on industrial pricing preferences and practices.

The idea of the relationship between demand, supply and price and its properties has also been highlighted in more recent papers. According to Berry and Haile (2021) the demand estimation is difficult because of endogeneity of prices and the fact that the demand for any good generally depends on more than one latent demand shock.

Babai, Boylan, and Rostami-Tabar (2022) analysed the issue of the demand forecast, which is the basis for most decisions in supply-chain management. They point out that the most accurate forecasts are not always obtained from data at the "natural" level of aggregation. In some cases, the forecast accuracy can be improved by aggregating data or forecasts at lower levels, disaggregating data or forecasts at higher levels, or by combining forecasts at multiple levels of aggregation.

This work provides a comprehensive review of research on aggregation and hierarchical forecasting in supply chains, including aggregation over time and other dimensions and combinations of forecasts at hierarchical levels. Perez and Lopez-Osima (2022) propose a model describing a non-linear profit maximization probabilistic problem. They deal with the nonlinearity and the multiple optima to solve their problem using an equivalent non-linear model and a particle swarm optimization (PSO) heuristic.

They find the prices that achieve equilibrium for the game. The authors additionally analyse the case from a Chilean telecommunications company and show the results regarding price decisions and market share effects. Babaeinesami, Ghasemi, Chobar, Sasouli, and Lajevardi, (2022) apply machine learning techniques to find the demand and supply forecast. They illustrate the proposed approach for historical data from 2005 to 2023 to predict the future demand in the automotive industry.

The machine learning model is then used to forecast the demand under different scenarios, including scenarios with supply chain disruptions. The paper also presents a case study of how the proposed approach was used to predict the demand for automotive components during the COVID-19 pandemic. The case study shows that the proposed approach was able to generate more accurate forecasts than traditional demand forecasting methods.

3. Research Methods

Let $CS_i(t)$ denote the surplus of a consumer i when buying a product (note the purchase event as $B = 1$) at time t. Assume that it depends on the price of the product $P(t)$ at time t and the difference of price over time which is represented by the derivative $P'(t)$. The surplus strongly changes from positive to negative when the price of the product breaks a barrier assumed by the consumer. The barrier may also be related to the financial resources assumed by the consumer.

This barrier should be the maximum price that is in line with consumer's expectations without any speculation. Let's denote this consumer-specific value by $P_i^{(\max)}$ for the i-th consumer. The above assumption represents a new approach to the consumer behaviour compared to Wang *et al.* (2007), Dost and Wilken (2012) and Dost *et al.* (2014) as:

- the dynamic component has been added,
- the consumers may purchase the product even if their surpluses are temporarily negative as some of them may be risk-loving rather than risk-averse.

Table 1 shows how the two factors, i.e. the price derivative over time, $P'(t)$, and the difference between the barrier and the price, i.e. $P_i^{(\max)} - P(t)$ affect the consumer's willingness to buy a product. The consumer is most willing to buy a product (high willingness) when the price is rising and is still below the $P_i^{(\max)}$.

Table 1*. The consumer's willingness to buy a product as a function of $P'(t)$ and $P_i^{(max)} - P(t)$*

Consumer's willingness to buy		$P_i^{(\max)} - P(t)$	
		< 0	> 0
$P'(t)$	> 0	**very low**	**high**
	< 0	**low**	**medium**

Source: *Own analysis.*

This is because a product gains in price over time and the price is below the assumed barrier. Certainly, the lower the price, the greater the willingness. The willingness is "medium" rather than "high" when the price is falling but is still below the $P_i^{(\max)}$. $P(t)$ lower than $P_i^{(\max)}$ represents a usual situation in the market. But when the prices exceeds the barrier $P_i^{(\max)}$, the consumer behaviour changes significantly.

The most important assumption is that some consumers are still willing to buy a product. However, they are less willing to buy a product when the price rises (very low willingness) compared to the case of the price fall (low willingness). The explanation is the following. When the price exceeds $P_i^{(\max)}$, and still rises, consumers expect that it will soon decline sharply. That is why they reduce the volume of purchases.

However, when the price exceeds $P_i^{(\max)}$, and declines, consumers believe that it will soon reach its normal level, i.e. below $P_i^{(\max)}$. In that case they are not as unwilling to buy a product as in the previous situation. The description of the four levels of willingness to buy is presented in Table 2.

Table 2. *The consumer's willingness to buy and the consumer surplus* $CS_i(t)$

The willingness to buy	$P_i^{(\max)} - P(t)$	$P'(t)$	$CS_i(t)$
highly undesirable	negative (price above the maximum)	positive (price increases)	The surplus is negative and this is the worst situation in the market from the consumer's perspective.
undesirable	negative (price above the barrier)	negative (price decreases)	Surplus is negative, but not as negative as in the "highly undesirable" case. This is because the price gradually approaches $P_i^{(\max)}$.
neutral	positive (price below the maximum)	negative (price decreases)	Surplus is positive, but it is not the most welcome situation for consumers as some of them refrain from buying a good if the price declines.
desirable	positive (price below the maximum)	positive (price increases)	Surplus is positive and this is the most welcome situation in the market from the consumer's perspective.

Source: *Own analysis.*

Let's assume additionally that price $P(t)$ function is continuous and is differentiable at time t. Let's also define the surplus $CS_i(t)$ at time t according to all above assumptions and properties in the following general form:

$$CS_i(t) = a_1 \cdot \left(P_i^{(\max)} - P(t)\right) \cdot e^{a_2 P'(t)} \tag{1}$$

where: $a_1 > 0$ is the scale parameter,
$a_2 > 0$ is the change dynamics parameter,
$P(t)$ the price of the product,
$P'(t)$ the trend of changing the price,
$P_i^{(\max)}$ the maximum price resulting from the i-th consumer's expectations, i.e. the maximum non-speculative value (let's call it the price barrier).

After taking the natural logarithm on both sides we get the following form of the above equation:

$$\ln\left(CS_i(t)\right) = ln(a_1) + ln\left(P_i^{(\max)} - P(t)\right) + a_2 P'(t) \tag{2}$$

We consider only one kind of assets. Therefore no comparison is made between different surplus functions and without loss of generality we can assume that the intercept in the above equation is zero and therefore $a_1 = 1$. Then the surplus considered takes the following form:

$$CS_i(t) = \left(P_i^{(\max)} - P(t)\right) \cdot e^{a_2 P'(t)} \tag{3}$$

The exponential function ensures that assuming the invariance between the price and maximum price the following conditions are conditionally met:

- the surplus is positively correlated with the trend of the price change (that is $P'(t)$) for the price lower than the barrier,
- the surplus is negatively correlated with the trend of price change for the price greater than the barrier,
- the surplus tends to move faster when the price rises by a unit than when it falls by a unit. This results from the properties of the exponential function. Intuitively, this happens because the consumer's willingness to buy a good increases faster due to the prospect of a good purchase than decreases due to a market collapse. This implicit assumption is made in the paper by defining the above formula for the surplus.

From the producers' perspective, the price can be determined in the following way:

$$P(t) = C + 2 \cdot M_{max} \cdot \left(1 - \frac{1}{\pi(B=1|t)+1}\right) \tag{4}$$

where: $0 < C < \infty$ is the marginal cost of production; let's assume it is constant over time:

$\pi(B = 1|t)$ is the expected demand in the population and can be interpreted as the probability to buy the product in the market by a random consumer,
$0 < M_{max} < \infty$ is the maximum margin of the product over the marginal cost that can be set by the producer and assume that it is constant in time.

From the formula above it can be seen that the price of the product set by the producer is greater if the marginal cost is higher or the average probability to buy the product in the market is higher. In the extreme case if the probability to buy the product in the market is zero, then the price equals the marginal cost. This equation shows that the price $P(t)$ belongs to the interval $P(t) \in [C, C + M_{max}]$. The price defined above is minimum equal to C when average demand is zero and maximum equal to $C + M_{max}$ when average demand is 100%.

Now we can set the following range of values for the surplus.

$$CS_i(t) \in [-\left(C + M_{max} - P_i^{(\max)}\right) \cdot e^{a_2 P'(t)}, -\left(C - P_i^{(\max)}\right) \cdot e^{a_2 P'(t)}] \tag{5}$$

Finally, let's assume that a consumer purchases a product at time t when the surplus $CS_i(t)$ exceeds any specific cut-off c_i, that is when $\boldsymbol{CS_i(t) > c_i}$. If $c_i > 0$ then the cut-off point is associated with positive surplus and then a consumer has risk aversion, otherwise consumer accepts a greater risk. In general lower value of c_i indicates greater exposure to risk.

There is also a group of consumers who are not interested in buying the product and then the cut-off is greater than the maximum value of the surplus. The higher cut-off for the consumer, the lower willingness to buy the product (*ceteris paribus*). Let's assume that the cut-off for this group can be infinitely large.

In summary we introduced the following portfolio quantities:

- three functions changing over time: price $P(t)$, price trend $P'(t)$ and probability to by the product $\pi(B = 1|t)$,
- two constant values over time: maximum producer's margin P_{max}, production costs C,
- an unknown parameter $a_2 > 0$,

and the following consumer specific quantities:

- consumer's price barrier $P_i^{(max)}$, cut-off for the willingness c_i and surplus function $CS_i(t)$.

Taking into account the whole population of consumers we can define a random variable c which has a distribution based on how all consumer's cut-offs are distributed. Let's notice that cut-off $\boldsymbol{c}$ can be both negative and positive, but its values in the population are definitely positive. Hence, its support is from the range $[c_{min}, \infty)$ where $c_{min} \leq 0$.

The specific consumer information such as barrier $P_i^{(\max)}$ is also changing in the population and can be represented by random variable $\boldsymbol{P^{(\max)}}$ and consequently the consumer's surplus $CS_i(t)$ can be represented in the population by random variable $\boldsymbol{CS(t)}$. On the other hand price $P(t)$ and its derivative $P'(t)$ are deterministic functions at time t that depend on aggregate consumer-specific information from the entire population. In summary, based on the entire consumer population we define three random variables that represent the cut-off for the willingness function $\boldsymbol{c}$, price barrier $\boldsymbol{P^{(\max)}}$ and surplus at time t, namely $\boldsymbol{CS(t)}$. Later in the article we will skip the index *i* because from now on we will be referring to random variables rather than to information about specific customers.

Taking into account the previous assumptions the probability π of buying the product by a random consumer is equal to:

$$\pi(B = 1|t) = P(CS(t) > c|t) \tag{6}$$

Let's consider the following exponential densities for random variables c and $P^{(\max)}$.

$$f(c) = \lambda \cdot e^{-\lambda \cdot (c - c_{min})}, c \geq c_{min}, \; c_{min} \leq 0, \lambda > 0 \tag{7}$$

Additionally assume that c_{min} is define in such a way that for all consumers in the population their surplus cannot be lower than the minimum cut-off, hence $CS(t) \geq c_{min}$. The consequence of this assumption is at least one consumer (with the cut-off c_{min}) who is always willing to purchase the product regardless the market situation. At the same time, this consumer or group of consumers are not separated from the rest of the population because we assume the continuous density for the cut-off $\boldsymbol{c}$.

$$f\left(P^{(\max)}\right) = \mu \cdot e^{-\mu \cdot \left(P^{(max)} - P_{min}\right)}, P^{(max)} \geq P_{min} \geq C, \; \mu > 0 \tag{8}$$

Note that P_{min} is the minimum value of the barrier in the population and c_{min} is the minimum surplus cut-off in the population. Based on the introduced markings and statistical assumptions we can propose some theoretical results concern the properties of the probability π of buying the product by a random consumer and the equilibrium price.

4. Theoretical Results

The following proposition allow us to transform of the probability $\pi(B = 1|t)$ and find its direct formula.

Proposition 1:

$$\pi(B = 1|t) = 1 - \mu \cdot \frac{e^{-\lambda \cdot e^{a_2 P'(t)} \cdot \left(P_{min} - P(t)\right) + \lambda \cdot c_{\min}}}{\mu + \lambda \cdot e^{a_2 P'(t)}} \tag{9}$$

Proof:
Assume as previously that $CS(t) \geq c_{min}$ for all consumers, and consequently from equation (3) that $\left(P^{(max)} - P(t)\right) \cdot e^{a_2 P'(t)} \geq c_{min}$. Then we can simplify the probability $\pi(B = 1|t)$ as follows:

$$\pi(B = 1|t) = P(CS(t) > c \mid t) = P\left(\left(P^{(max)} - P(t)\right) \cdot e^{a_2 P'(t)} > c \mid t\right) =$$

$$= \int_{-\infty}^{\infty} \mathrm{P}\left(\left(P^{(max)} - P(t)\right) \cdot e^{a_2 P'(t)} > c \mid P^{(\max)}, t\right) \cdot f\left(P^{(\max)}\right) dP^{(\max)} =$$

$$= \int_{P_{min}}^{\infty} \left(\int_{c_{min}}^{\left(P^{(max)}-P(t)\right)\cdot e^{a_2 P'(t)}} \lambda \cdot e^{-\lambda\cdot(c-c_{min})}\, \mathrm{dc} \right) \cdot \mu \cdot e^{-\mu\cdot\left(P^{(max)}-P_{min}\right)} dP^{(\max)}$$

$$= \int_{P_{min}}^{\infty} \left(1 - e^{-\lambda\cdot\left(\left(P^{(max)}-P(t)\right)\cdot e^{a_2 P'(t)}-c_{min}\right)}\right) \cdot \mu \cdot e^{-\mu\cdot\left(P^{(max)}-P_{min}\right)} dP^{(\max)}$$

$$= 1 - \mu \cdot e^{-\lambda\cdot\left(-P(t)\cdot e^{a_2 P'(t)}-c_{min}\right)+\mu\cdot P_{min}} \cdot \int_{P_{min}}^{\infty} e^{-P^{(\max)}\cdot\left(\mu+\lambda\cdot e^{a_2 P'(t)}\right)} dP^{(\max)}$$

$$= 1 - \mu \cdot e^{-\lambda\cdot\left(-P(t)\cdot e^{a_2 P'(t)}-c_{min}\right)+\mu\cdot P_{min}} \cdot \frac{1}{\mu + \lambda \cdot e^{a_2 P'(t)}} \cdot e^{-\left(\mu+\lambda\cdot e^{a_2 P'(t)}\right)P_{min}}$$

$$= 1 - \mu \cdot \frac{e^{-\lambda\cdot e^{a_2 P'(t)}\cdot\left(P_{min}-P(t)\right)+\lambda\cdot c_{\min}}}{\mu+\lambda\cdot e^{a_2 P'(t)}},$$

what need to be proven.

Combining equations (4) and (9) we have:

$$\frac{1}{\frac{2\cdot M_{max}}{P(t)-C}-1} = 1 - \mu \cdot \frac{e^{-\lambda\cdot e^{a_2 P'(t)}\cdot\left(P_{min}-P(t)\right)+\lambda\cdot c_{\min}}}{\mu+\lambda\cdot e^{a_2 P'(t)}} \tag{10}$$

In order to find the equilibrium point we need to examine the properties of price $P(t)$ involved in equation (10). Investigation the monotonicity of the price $P(t)$ on the basis of its derivative $P'(t)$ is not an easy task due to a more complex formula above. In general, the price $P(t)$ is increasing when its derivative is positive $P'(t) > 0$, and decreases otherwise. The following lemma is the auxiliary to find the equilibrium point and it assumes that the derivative of price $P(t)$ is zero $P'(t) = 0$.

Lemma 1:
There exists a unique solution $P_0(\mu, \lambda, P_{min}, c_{min}, M_{max}, C) \in [C, C + M_{max}]$ (abbreviated as P_0) of the following equation with respect to variable P:

$$\frac{P-M_{max}-C}{P-2\cdot M_{max}-C} = d \cdot e^{\lambda\cdot P} \tag{11}$$

where: $d = \frac{\mu}{2(\mu+\lambda)} \cdot e^{-\lambda\cdot P_{min}+\lambda\cdot c_{\min}}$.

Proof:

Let's assume that $P'(t) = 0$. Then after some basic transformations, equation (10) can be written as:

$$\frac{P(t) - M_{max} - C}{P(t) - 2 \cdot M_{max} - C} = d \cdot e^{\lambda \cdot P(t)} \tag{12}$$

where: $d = \frac{\mu}{2(\mu+\lambda)} \cdot e^{-\lambda \cdot P_{min} + \lambda \cdot c_{\min}}$.

Taking into account that $c_{min} \le 0$ and $P_{min} > 0$ we get that $0 < d < \frac{1}{2}$.

We can find the approximate solution of the equation (12) by plotting the two curves in the whole range of $P(t) \in [C, C + M_{max}]$ related to the left and right side of equation (12) and analysing their properties. Let's denote the functions defining the curves by $f_1(P(t))$ and $f_2(P(t))$. Note that we are analysing properties of these two functions in relation to argument $P = P(t)$ but not the time t.

$$f_1(P(t)) = \frac{P(t) - M_{max} - C}{P(t) - 2 \cdot M_{max} - C}, \quad f_2(P(t)) = d \cdot e^{\lambda \cdot P(t)} \tag{13}$$

Let's notice that function $f_1(P(t))$ has the following mathematical properties:

- it tends to value 1 in minus infinity,
- it has vertical asymptote at value $2M_{max} + C$,
- $f_1(M_{max} + C) = 0$,
- $f_1(C) = \frac{1}{2}$.

Some of them are only mathematical properties outside the acceptable range of the price.

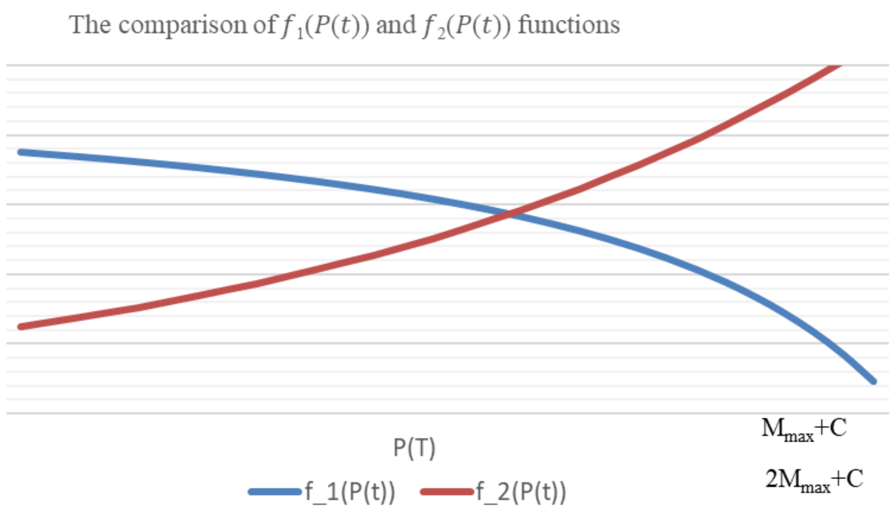
Note that $f_2(P(t))$ is an increasing function which has the following mathematical properties:

- $f_2(0) = d \left(0 < d < \frac{1}{2}\right)$,
- $f_2(C) < \frac{1}{2}$ since $P_{min} \ge C$ and $c_{min} \le 0$.

The approximate course of the curves can be presented on the basis of the properties above (**Σφάλμα! Το αρχείο προέλευσης της αναφοράς δεν βρέθηκε.**).

This is clear that curves for each time t have only one common point $P_0 > C$ independent on time t and equation (12) holds when $P(t) = P_0$. Point P_0 is dependent on constants $\mu, \lambda, P_{min}, c_{min}, M_{max}, C$ what need to be proven. Let's call equation (12) the price stability equation (PSE).

Figure 1. *Graphical solution of equation (12)*



Source: *Own analysis.*

It should be noted that the curves in Figure 1 show the left and right-hand sides on the PSE. Thus, they should not be considered demand and supply curves. When the price reaches P_0, the dynamically stable equilibrium is achieved.

Ultimately, we can formulate the following final rule in the proposition 2.

Proposition 2:

a. Function $P(t)$ has a unique limit when time goes to infinity.
b. Its limit is equal to P_0

$$\lim_{t\to\infty} P(t) = P_0, \tag{14}$$

c. the demand represented by the probability $\pi(B = 1|t)$ tends to value $\pi_\infty(M_{max}, C)$.

$$\pi_\infty(M_{max}, C) = \frac{1}{\frac{2\cdot M_{max}}{P_0 - C} - 1} \tag{15}$$

Proof:

In order to prove the properties a. and b. recall that equation (4) shows that price $P(t)$ is limited by the finite interval $P(t) \in [C, C + M_{max}]$, when $M_{max} < \infty, C < \infty$.

This part of the proposition can be proven using an implicit proof. Imagine that there are at least two different subsequences for which $P(t)$ tends to two different values.

Then the derivative $P'(t)$ tends to zero for each of these two subsequences and equation (10) appears to be (12). From the continuity of function $P(t)$ and proven Proposition 1, the limit of function $P(t)$ for each subsequence satisfies equation (12) and consequently is equal to P_0,what denies the primary assumption about at least two different subsequences for which $P(t)$ tends to two different values and consequently proves the proposition thesis a and b.

The third part of the proposition can be proven by substituting value P_0 into equation (4). From theses a. and b. we know that price $P(t)$ has a unique limit equal to P_0, which proves that the demand represented by the probability $\pi(B = 1|t)$ tends to a certain value π_∞ that satisfies equation (15).

The last proposition ensures us that price $P(t)$ tends to its unique value P_0 only monotonically.

Proposition 3:

Function $P(t)$ increases when $P(t) < P_0$, otherwise decreases.

Proof:

The thesis of the proposition is satisfied since only a monotonic function can satisfy the following properties of price $P(t)$.

- function $P(t)$ is a continuous function,
- P_0 is the only value for which price $P(t)$ has a zero derivative what was proven in Proposition 1,
- $\lim_{t\to\infty} P(t) = P_0$ according to Proposition 2.

5. Empirical results

To apply the rule between the demand and the price found in the previous section we have to realize that two random variables ($\boldsymbol{c}$ and $\boldsymbol{P}^{(max)}$) are being observed and set the following constants that determines the final results:

$\boldsymbol{a_2} > 0$ – coefficient in the surplus formula,
$\boldsymbol{C} > 0$ – marginal cost,
$\boldsymbol{M_{max}} > 0$ – maximum possible margin of the product,

$\boldsymbol{c}$ is the cut-off of the willingness to buy a product that has an exponential distribution over the population,

$\boldsymbol{c} \in [\boldsymbol{c_{min}}, \infty)$,
$\boldsymbol{c_{min}} \leq 0$ – minimum value of cut-off $\boldsymbol{c}$ in the population,

$\boldsymbol{\lambda} > 0$ – parameter in the distribution of the variable c, that is $E(c) = \frac{1}{\lambda}$,

$\boldsymbol{P}^{(max)}$ is the price barrier with exponential distribution over the population,

$\boldsymbol{P}^{(max)} \in [\boldsymbol{P}_{min}, \infty)$,
$\boldsymbol{\mu} > 0$ – parameter in the distribution of the variable $P^{(\max)}$, that is $E(P^{max}) = \frac{1}{\mu}$,
$\boldsymbol{P}_{min} \in [\boldsymbol{C}, \boldsymbol{C} + \boldsymbol{M}_{max}]$ – the minimum price barrier in the population.

All deterministic parameters from the list above can be found by analysing historical data except for the coefficient in the surplus formula. This coefficient could be found by analysing consumers' behaviour and their willingness to buy a product in different circumstances. This is not as easy a task as simple historical data analysis.

Let's consider an example of the real estate market in 4Q 2022 in Poland. Some of the assumptions below are based on expert opinion, and all calculations below are a simple demonstration of the application of the proposed approach to find price P_0.

Marginal costs represent the most fundamental information and other parameters are dependent on them. The price of one square meter of usable floor space of a residential building in 4Q 2022 in Poland was $C = 5{,}708$. (GUS 2023). Let's further assume that:

- The maximum producer's margin is twice the marginal cost, i.e. $M_{max} = 2 \cdot C$. Hence, the maximum possible price is equal to $\max\{P(t)\} = M_{max} + C = 3 \cdot C$.
- The minimum value of the price barrier (P_{min}) can be assigned to minimum price which is equal to the marginal cost, i.e. $P_{min} = C$. This means that there are consumers in the population who are very strongly risk-averse or are not willing to purchase a given good.
- average accepted barrier ($E(P^{max})$) is equal to twice the marginal cost, i.e. $E\left(\boldsymbol{P}^{(max)}\right) = \frac{1}{\mu} = 2 \cdot C \;\Rightarrow\; \mu = \frac{1}{2 \cdot C}$.
- The most elusive value is the average cut-off in the population. In 2022 a housing boom generated by the availability of mortgage products slowed down according to rising interest rates and the average cut-off in the population also increased. The current real estate market shows that prices do not change significantly. We can assume that the average cut-off ($E(\boldsymbol{c})$) is lower than the average barrier price ($P^{(max)}$) and is equal to $\frac{1}{2} \cdot E\left(\boldsymbol{P}^{(max)}\right) = C$. Then we have
$E(\boldsymbol{c}) = \frac{1}{\lambda} = C \;\Rightarrow\; \lambda = \frac{1}{C}$.

- Let's assume ultimately that $c_{min} = -\frac{C}{3}$, what means that if surplus exceeds this value, i.e., a random consumer buys the product.

$c_{min} = -\frac{C}{3}$, $\min\{P(t)\} = P_{min} = C$, $E\left(\boldsymbol{P}^{(max)}\right) = M_{max} = 2 \cdot C$, $3 \cdot C = \max\{P(t)\}$

Taking into account all the assumptions above and substituting all constants into equation (12) we get:

$$\frac{P-2C-C}{P-4C-C} = d \cdot e^{\frac{1}{C} \cdot P} \tag{16}$$

where: $d = \frac{\frac{1}{2C}}{2\left(\frac{1}{2C}+\frac{1}{C}\right)} \cdot e^{-\frac{1}{C} \cdot \left(C+\frac{C}{3}\right)} = \frac{1}{6} \cdot e^{-\frac{4}{3}}$

Finally, stability price $P_0 = 10.385$ is a unique solution of the PSE, what is illustrated in Figure 2.

$$\frac{P-3C}{P-5C} = \frac{1}{6} \cdot e^{\frac{1}{C} \cdot P - \frac{4}{3}} \tag{17}$$

6. Discussion

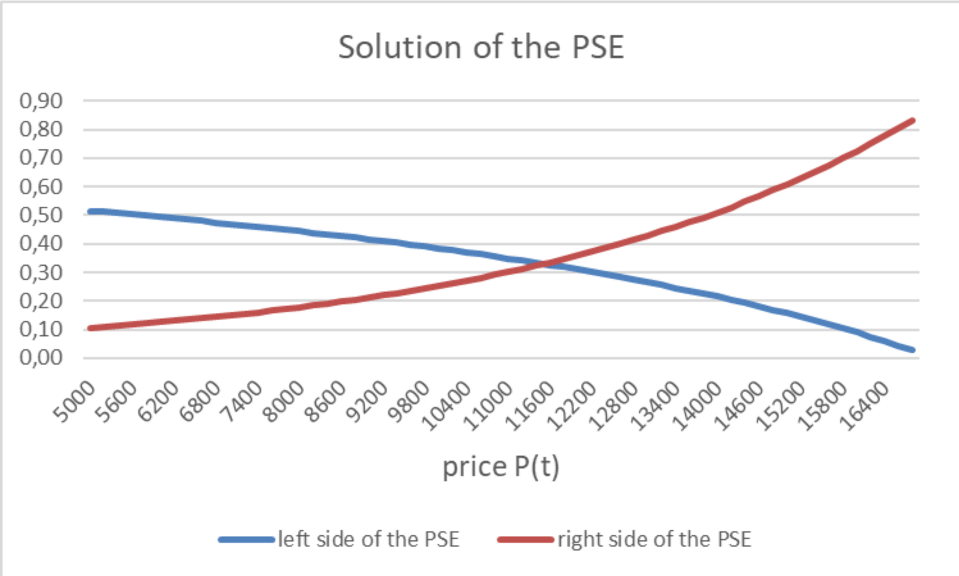
The method presented in the article is based on the statistical approach including the distributions of an exponential family. The distributions are assumed for the following variables: the cut-off for the customer willingness function, the customer price barrier and the consumer's surplus at time, namely.

The exponential family is a broad class of distributions and has good properties for obtaining direct theoretical results. Nevertheless, it would be worth discussing also empirical distributions or another class of the theoretical distributions of the mentioned random variables in more detail after an in-depth analysis of the consumer behaviour.

Our research study is designed for the mortgage market using exponential distribution family for all random variables introduced in the article.

This is a broad class of distributions that we believe can also be applied to other goods on the market. The proposed approach takes into account many economic variables, including: marginal cost, maximum possible margin, minimum value of cut-off c in the population, the minimum price barrier in the population and parameters in the exponential distributions.

Figure 2. *The solution of the equilibrium equation for the mortgage market in Poland in 2022*



Source: *Own analysis.*

Their precise estimation represents a challenge, therefore our approach is to define all of them on the basis of one parameter, i.e., the marginal cost. The results are new and different than the ones obtained by Chakravarty (1999), as well as Kumar *et al.* (2009) due to distinct approaches considering the utility functions. Moreover, the findings are not the same as the ones obtained by Wang (2007) and Dost an Wilken (2012) as different assumptions regarding the consumer surplus have been applied.

Nevertheless, our method of mathematical modelling is broadly similar in spite of the existing divergencies in case of conclusions. The proposed statistical procedure may have more applications. i.e. take into account other markets and goods' characteristics to obtain more accurate fluctuations of the price over time.

Moreover, it would be valuable to further analyse the consumer's willingness to pay and seek other parameters that might affect the consumer's surplus. It is also worth mentioning that our general implications concerning the economic practice are comparable to those formulated by Lizou et al. (2012). Our approach is similar to those suggested by Schulze et al., where the theoretical framework is flexible and can be applied to different models as well.

7. Conclusions

In the paper we made an assumption on random behaviour of the consumers in terms of their willingness to purchase the product and their surpluses. We also assumed that the consumer could buy the good even if his or her surplus was temporarily negative. Moreover, that surplus also depends on whether the price increases or decreases over time.

Nevertheless, the producer's behaviour in terms of pricing is deterministic and based on the aggregate information from the market. We applied the exponential family distributions for the two random variables: the cut-off $\boldsymbol{c}$, i.e. the point at which a consumer is on the verge of making a decision to purchase the product and $\boldsymbol{P}^{(max)}$, i.e. a barrier of the price that is not speculative in nature.

We introduced a consumer's surplus function $\boldsymbol{CS}(\boldsymbol{t})$ at time t, which is related to the willingness of the consumer to buy the product and we combined this function with the probability of buying the product $\boldsymbol{\pi}(\boldsymbol{B} = \mathbf{1}|\boldsymbol{t})$ at time t.

On grounds of the relationship between the surplus function and the probability of purchasing the product several theorems were proven. They describe the properties of the price change over time and determine the dynamic stability of the price equilibrium point. The exact value of the equilibrium price represents an implicit solution of the selected mathematical equation, which is not known in closed form.

We applied the derived rule to find the dynamically stable price for the Polish mortgage market in 4Q 2022. The final price is based on some constants that represent a function of the marginal cost. The proposed rule is general and can be applied to other goods and markets.

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