
Fertilizer Purchase Optimization as a Problem of Mixed Integer Nonlinear Programming

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Abstract:

Purpose: This paper aims to introduce a model of fertilizer purchase optimization - an improvement of the one originally developed for supporting African farmers. The improvement takes into account necessity of purchasing fertilizers in bags of fixed weight instead of arbitrary amounts.

Design/Methodology/Approach: A fertilizer purchase optimization model expressed as a nonlinear programming problem and its implementation in Microsoft Excel, once developed for a project named Optimized Fertilizer Recommendations in Africa (OFRA), were analysed. An extension of the above model in the form a mixed integer nonlinear programming problem and its implementation in Microsoft Excel were developed.

Findings: The model of fertilizer purchase optimization developed for OFRA omits an important issue – availability of fertilizers in fixed-sized “portions” only (50 kg bags). An improved model which includes the inevitable purchases of fixed-sized “portions” of fertilizers into the optimality criterion is introduced.

Practical Implications: The improved model is much more compliant with the conditions of the fertilizer market than the original one whereas performing the optimization remains unchanged from the point of view of the user.

Originality/Value: Creating a fertilizer purchase optimization model taking into account real market conditions (sale of fertilizers in fixed-sized “portions”) handles an issue which is disregarded in many existing models despite its influence on the final financial output.

Keywords: Fertilizers, profit maximization, nonlinear programming, integer programming.

JEL codes: C61, N57, O13.

Paper Type: Research paper.

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1. Introduction

The usage of fertilizers can be, like many other fields of human activity, a subject of optimization. For the purpose of this paper, we consider *fertilizers* as chemical substances (single chemical compounds or blends of compounds). Chemical elements contained in fertilizers are called *nutrients*. The simplest approach to optimization of fertilizer usage is to minimize the purchase cost of a mix of fertilizers regarding norms of supplies of nutrients. Basically, it is just as a variant of the optimal diet problem, dating back to 1940's (Stigler, 1945). A unit surface of cultivated land (usually 1 hectare or 1 acre) is “fed” with fertilizers as just as a single human being or an animal. Target amounts of nutrients which must be applied to this unit surface of land depend on the kind of a crop and natural conditions (soil and climate). They are specified as lower limits (optionally together with upper limits) or exact values (Minh, 2007; Bueno-Delgado *et al.*, 2016). They can also be defined as relative amounts (percentages) with no connection to any specific surface. In this case a target amount of the blend of fertilizers per hectare/acre must also be specified (Aldeseit, 2014). The purchase cost minimization is not the only criterion of optimality – e.g., maximization of profit or amount of nutrient is also possible (Minh, 2007).

More precise modelling of fertilizer usage can include uncertainty of parameters of the models (Jareonkitpoolpol *et al.*, 2018) or availability of fertilizers in fixed-sized “portions” only (Glover, 1977; Atesmen, 2011). The latter issue requires some explanations. Whereas fertilizers are divisible (they are powdered/granulated solids or liquids), they do not need to be sold in any required amounts. Instead, they are sold in “portions” resulting from packaging (e.g., bags of fixed net weight). This means that modelling amounts of fertilizers as real-valued variables is not correct when refers to amounts of purchased fertilizers. Some integer variables must then be added to the model to reflect the real-world conditions correctly.

Another important modelling issue is connected with specifying target values for amounts of nutrients. Whereas it is obvious that applying large amounts of fertilizers, and, what follows, nutrients can be potentially harmful for the crops, it is worth considering what will happen to the yield if the amounts of applied fertilizers change from zero to some “safe” upper limits on amounts of nutrients. One of possibilities is using goal programming (Mínguez *et al.*, 1988). If relationships between amounts of nutrients delivered to the crops in the applied fertilizer and the resulting yield can be expressed as a function with a “simple” formula, then such function can be included in an optimization model (Jansen *et al.*, 2013; Rware *et al.*, 2016).

In this paper a profit-maximization approach with nutrient amount-yield dependence and purchase of fertilizers in fixed-sized “portions” is presented. It is an extension of an optimization model developed for a project named Optimized Fertilizer Recommendations in Africa (OFRA), developed with scientific support from the

University of Nebraska–Lincoln, USA. An important part of the OFRA project was developing so-called *fertilizer optimization tools* (FOTs) to make the theoretical model applicable to farmers. FOTs are software tools. One is an Android mobile application and another is a Microsoft Excel-based application using simple user interface created in Visual Basic and a built-in Solver optimization add-in. The output of FOTs is a solution of some nonlinear programming problem which is assumed to be an optimal (profit-maximizing) allocation of purchased fertilizers among crops subject to the budget constraint and upper limits of amounts of nutrients and fertilizers (Kaizzi *et al.*, 2017). However, both the theoretical model and its software implementation are missing a very important condition of the fertilizer market, namely the necessity of the purchase of fertilizers in fixed-sized “portions” (50 kg bags). This fact essentially affects the final financial output of the agricultural activity but it was not addressed in the OFRA model. This paper offers a correction of this drawback without replacing Microsoft Excel with another software.

2. A Fertilizer Optimization Model and Its Extension

The main idea standing behind the model under consideration is that requirement for each nutrient is not a fixed number depending on the type of crop and local natural conditions like soil or climate. Instead, a so-called *crop nutrient response function* is used. Such a function describes how the amount of an applied nutrient affects the “response” i.e., the amount of the resulting yield of the crop. In case of the OFRA research, crop nutrient response function were determined from results of field research conducted in 67 prime agricultural ecozones (AEZs) in 13 Sub-Saharan countries (Kaizzi *et al.*, 2017). Those functions were defined as asymptotic curvilinear-plateau functions taking the form of an exponential rise to a maximum or plateau yield. The general formula for such a kind of function is:

$$Y(z) = a - bc^z$$

where:

- $Y(z)$ is yield (t/ha),
- $a, a > 0$ is the maximum or plateau yield (t/ha) for application of a specific nutrient,
- $b, b > 0$ is the maximum gain in yield (t/ha) due to application of the nutrient,
- $c, 0 < c < 1$ is a curvature coefficient (c^z represents the shape of the response function),
- z is the nutrient application rate (kg/ha).

If no nutrient is applied then $Y(0) = a - bc^0 = a - b$ (the minimal possible yield). The increase of yield after applying the amount of the nutrient equal to z is $Y(z) - Y(0) = a - bc^z - (a - bc^0) = a - bc^z - a + bc^0 = b - bc^z$.

It turns out that the increase of the yield, and, what follows, the revenue depends on the b and c parameters only. In the “full” optimization model b , c and z are supplemented with pairs of indices (where the first one denotes the number of the crop, the second one the number of the nutrient). If some crop does not show a “response” to a specific nutrient, then $b = 0$ and $c = 1$. Setting $c = 0$ may be troubling for some optimization software because it may lead to the undefined expression 0^0 .

The following model was created within the OFRA project. There are m kinds of crops, n kinds of fertilizers and p nutrients. The following parameters are given:

- $p_i, i = 1, 2, \dots, m$ – selling price of 1 kg of crop i ;
- $a_i, i = 1, 2, \dots, m$ – area in hectares planted by crop i ;
- $c_j, j = 1, 2, \dots, n$ – purchase price of 1 kg of fertilizer j ;
- B – maximal amount of money available for purchase of fertilizers (the budget limit);
- $P_{jk}, j = 1, 2, \dots, n, k = 1, 2, \dots, p$ – percentage of nutrient k in fertilizer j ;
- $F_{ij}, i = 1, 2, \dots, m, j = 1, 2, \dots, n$ – maximal amount of fertilizer j in kg which can be applied to 1 hectare of crop i ;
- $N_{ik}, i = 1, 2, \dots, m, k = 1, 2, \dots, p$ – maximal amount of nutrient k in kg which can be applied to 1 hectare of crop i ;
- $b_{ik}, c_{ik}, i = 1, 2, \dots, m, k = 1, 2, \dots, p$ – coefficients which describe the response of the volume of yield of crop i depending on the amount of nutrient k (parameters of crop nutrient response functions for each pair crop-nutrient)
Variables describe allocation of fertilizers among the crops.
- $x_{ij}, i = 1, 2, \dots, m, j = 1, 2, \dots, n$ – amount of fertilizer j which is applied to 1 hectare of crop i .

For simplicity of the further notation, we introduce the following auxiliary variables:

- $z_{ik}, i = 1, 2, \dots, m, k = 1, 2, \dots, p$ – amount of nutrient k which is applied to 1 hectare of crop i .

The objective function is the profit – the difference between revenue resulting from the increase of yield after applying fertilizers to the crops and the cost of fertilizers. The constant 1000 is necessary to transform the outputs of $(b_{ik} - b_{ik} c_{ik}^{z_{ik}})$ from tons to kilograms. The entire optimization model is the following:

$$\sum_{i=1}^m \sum_{k=1}^p 1000 p_i a_i (b_{ik} - b_{ik} c_{ik}^{z_{ik}}) - \sum_{j=1}^n \sum_{i=1}^m c_j a_i x_{ij} \rightarrow \max$$

subject to the constraints

$$\begin{aligned}
 x_{ij} &\geq 0, i = 1, 2, \dots, m, j = 1, 2, \dots, n \\
 x_{ij} &\leq F_{ij}, i = 1, 2, \dots, m, j = 1, 2, \dots, n \\
 z_{ik} &= \sum_{j=1}^n P_{jk} x_{ij}, i = 1, 2, \dots, m, k = 1, 2, \dots, p \\
 z_{ik} &\leq N_{ik}, i = 1, 2, \dots, m, k = 1, 2, \dots, p \\
 \sum_{j=1}^n \sum_{i=1}^m c_j a_i x_{ij} &\leq B
 \end{aligned}$$

The most significant drawback of the above model itself and its Excel implementation as well is that it does not take into account an important feature of the distribution of fertilizers. Namely, they must be purchased in fixed-size “portions” because they are available in 50 kg bags only. This fact may seriously affect the financial output of the solution. A slight extension of the OFRA model allows to create a relatively simple workaround (described below) which does not guarantee obtaining an optimal solution, however.

Let us denote by x_{ij}^* ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) the optimal amount of fertilizer j which is applied to 1 hectare of crop i and by z_{ik}^* ($i = 1, 2, \dots, m, k = 1, 2, \dots, p$) the optimal amount of nutrient k which is applied to 1 hectare of crop i . Let

$$X_j^* = \sum_{i=1}^m a_i x_{ij}^*, j = 1, 2, \dots, n$$

be the optimal amount of fertilizer j and r_j^* the nearest multiple of 50 equal or greater than X_j^* , $j = 1, 2, \dots, n$ (e.g., for $X_j^* = 118.26$ $r_j^* = 150 = 3 \cdot 50$). The real price the farmer who wants to buy fertilizers in amounts X_j^* must pay is the price for amounts r_j^* , not X_j^* (the price of integer number of bags). An important question arises – what to do with excess amounts of fertilizers $r_j^* - X_j^*$? If there is no opportunity to resell them at the price of purchase or a slightly lower one, the best option seems to be to apply them to increase the yield what, thanks to higher revenues, will partially compensate the unwanted spending on excess amounts of fertilizers. But this idea is not that easy to implement as it may look like.

The total amount of each fertilizer must be distributed among various crops but the excess amounts should not be distributed proportionally to the values x_{ij}^* for each $j = 1, 2, \dots, n$. There are two reasons for the above statement. The first one is that dependencies of yields on amounts of nutrients are nonlinear. The second one is

that such “excess distribution” may result in exceeding limits on maximal amounts of nutrients or fertilizers. This is why distributing excess amounts must be included in the optimization problem. Having calculated r_j^* - optimal amounts of fertilizers rounded up to multiples of 50, the above optimization problem should be recalculated once again, but with added new constraints:

$$\sum_{i=1}^m a_i x_{ij} = r_j^*, j = 1, 2, \dots, n$$

which provide a correct distribution of all the purchased fertilizers among crops. Such a formulation will fail, however, if it is impossible to distribute fertilizers without exceeding at least one of the limits on maximal amounts of nutrients or fertilizers.

The only way to include the purchase of fertilizers sold in fixed net weight bags which guarantees optimality of the solution is to extend the above mathematical model. Let us introduce new integer variables:

- $y_j, j = 1, 2, \dots, n$ – number of 50 kg bags of fertilizer j .

They will appear in the cost component of the objective function instead of x_{ij} variables. Finally, a new model is described below. It uses all the parameters of the old model, new integer variables y_j together with x_{ij} real variables and z_{ik} auxiliary variables. The objective function is modified and some new constraints are added:

$$\sum_{i=1}^m \sum_{k=1}^p 1000 p_i a_i (b_{ik} - b_{ik} c_{ik}^{z_{ik}}) - \sum_{j=1}^n 50 c_j y_j \rightarrow \max$$

subject to the constraints

$$x_{ij} \geq 0, i = 1, 2, \dots, m, j = 1, 2, \dots, n$$

$$x_{ij} \leq F_{ij}, i = 1, 2, \dots, m, j = 1, 2, \dots, n$$

$$z_{ik} = \sum_{j=1}^n P_{jk} x_{ij}, i = 1, 2, \dots, m, k = 1, 2, \dots, p$$

$$z_{ik} \leq N_{ik}, i = 1, 2, \dots, m, k = 1, 2, \dots, p$$

$$\sum_{j=1}^n \sum_{i=1}^m c_j y_j \leq B$$

$$y_j \geq 0, j = 1, 2, \dots, n$$

$$y_j \text{ integer, } j = 1, 2, \dots, n$$

$$\sum_{i=1}^m a_i x_{ij} \leq 50y_j, j = 1, 2, \dots, n$$

The last group of constraint means that the total amount of fertilizer j distributed among all the crops cannot exceed some multiple of 50 kg. Because the yield and the revenue increase as the amounts of available fertilizers increase, it is probable that all the purchased fertilizers will be applied. But in some cases, less fertilizer than the multiple of 50 kg will be used. It can be caused by reaching the upper limits of amounts of fertilizers or nutrients

3. Results of Test Calculations

The data for the calculations were downloaded on 17th May 2021 from the OFRA section of the website of Department of Agronomy and Horticulture, University of Nebraska-Lincoln (<https://agronomy.unl.edu/OFRA>). The website stores many resources related to the OFRA project, including example fertilizer optimization tools (FOTs) as Excel files. Among them, the following file was chosen to serve as a source of the example input data and as a reference for comparing the output with the one of the model presented in Chapter 2. The file was „Benin June 2018.zip/Benin June 2018/Benin South Guinea FOT 18 June 2018.xlsm” which is stored under the address:
<https://unl.app.box.com/s/w3vzxiaty19r07duxpwf0qon0c6zj791/file/300621112866>.

The content of the file is a FOT with the following data: 9 kinds of crops, 5 kinds of fertilizers and 4 nutrients (NPKS: nitrogen, phosphorus, potassium, sulphur). The budget limit (maximal amount of money which can be spent on purchase of fertilizers) is 1 000 000 and the prices of the fertilizers per kg are 200, 300, 280, 320, 3000. All money-related parameters are presumably expressed in CFA francs, the currency of Benin. The areas of crops are 1 ha each, with exception of crop 7 which is 2 ha.

The file is provided with an already calculated solution. The optimal amounts of the fertilizers are $X_1^* = 941.26$, $X_2^* = 177.29$, $X_3^* = 630.45$, $X_4^* = 258.03$, $X_5^* = 22.86$, the revenue is $R^* = 15\,072\,484.18$, the cost of fertilizers is $C^* = 569\,125.26$ and the profit is $P^* = R^* - C^* = 14\,503\,358.92$. But obviously it is impossible to buy such amounts of the fertilizers since they are sold in 50 kg bags. The minimal possible purchase is $r_1^* = 950$, $r_2^* = 200$, $r_3^* = 650$, $r_4^* = 300$, $r_5^* = 50$. The real cost after rounding up amounts of the fertilizers to the multiples of 50 increases to $C_{50}^* = 678\,000$ so the profit decreases to $P_{50}^* = 14\,394\,484.18$. However, assumptions of the original model allow to use excess amounts of fertilizers. Then by adding the following constraints we can make amounts of the fertilizers be r_j^* 's (force applying all the purchased fertilizers):

$$\sum_{i=1}^m a_i x_{ij} = r_j^*, j = 1, 2, \dots, n$$

and then recalculate the problem. The recalculated revenue is $R_{50r}^* = 15\,137\,999.51$, the cost of fertilizers is $C_{50r}^* = C_{50}^* = 678\,000$ (it is the same since it was fixed on the previous level by fixing amounts of the fertilizers to be r_j^* 's) and finally, the recalculated total profit is $P_{50r}^* = 14\,459\,999.51$. So, the excess cost of obligatory purchase of fertilizers in 50 kg bags instead in arbitrary amounts is $P^* - P_{50r}^* = 43\,359.41$.

What is interesting, the solution provided in the file is not optimal. Tests showed that it is obtained by starting from zero values in variable cells. When Solver stops calculations, it displays the message “Solver has converged to the current solution. All Constraints are satisfied”. It means that the solution calculated by Solver is not an optimal but just a feasible solution. However, after recalculation of the initial problem, the solution improves and Solver terminates with the message “Solver found a solution. All Constraints and optimality conditions are satisfied”. In the meantime, the message “The maximum iteration limit was reached; continue anyway?” may display (due to computational complexity of the problem) and the “Continue” option should be chosen.

The above phenomenon of not finding an optimal may happen when Solver is used to solving nonlinear programming problems, even if they have a global maximum like the considered case. Even if the “Solver found a solution (...)” message is displayed, it is worth considering recalculate the problem again to check if the solution improves. New, now optimal amounts of the fertilizers are $X_1^* = 1007.63$, $X_2^* = 2.70$, $X_3^* = 779.65$, $X_4^* = 257.55$, $X_5^* = 22.35$, the revenue is $R^* = 15\,090\,645.47$, the cost of fertilizers is $C^* = 570\,100.17$ and the total profit is $P^* = 14\,520\,545.30$. Rounded amounts of the fertilizers are $r_1^* = 1050$, $r_2^* = 50$, $r_3^* = 800$, $r_4^* = 300$, $r_5^* = 50$, so the real cost of purchase is $C_{50}^* = 695\,000$ and the profit decreases to $P_{50}^* = 14\,395\,645.47$. After recalculating with amounts of fertilizers forced to be r_j^* 's, the revenue is $R_{50r}^* = 15\,137\,999.51$, and the recalculated total profit is $P_{50r}^* = 14\,465\,042.66$. The excess cost of obligatory purchase of fertilizers in 50 kg bags instead in arbitrary amounts is $P^* - P_{50r}^* = 55\,502.41$.

Next, the data from the original file were used to test a new mathematical model in which purchasing fertilizers in 50 kg “portions” is directly included in the optimality criterion. The optimal amounts of the fertilizers are $X_1^* = 900$, $X_2^* = 50$, $X_3^* = 700$, $X_4^* = 250$, $X_5^* = 50$, the revenue is $R^* = 15\,084\,565.86$, the cost of fertilizers is $C^* = 621\,000$ and the total profit is $P^* = 14\,463\,565.86$. Solver terminated with

the message “Solver found an integer solution within tolerance. All Constraints are satisfied”.

This message displays in Excel 2010 or newer if at least one variable in the Solver model is set to be integer. It is related with the Solver option named “Integer Optimality (%)” which allows to terminate solving an optimization problem with integer variables without attaining optimality. The larger is value of this option, the more the time of calculations is preferred to the optimality of a solution. The default value of this option in Excel 2010 or newer is 1%. An optimal solution is assumed to be found with the value of the option equal to 0%.

This feature is implemented to speed up possibly very time-consuming integer optimization calculations by sacrificing optimality in favour of shorter time of calculations. After recalculation with “Integer Optimality (%) set to 0, the solution is $X_1^* = 1000$, $X_2^* = 50$, $X_3^* = 750$, $X_4^* = 250$, $X_5^* = 50$. Finally, the revenue is $R^* = 15\,125\,151.95$, the cost of fertilizers is $C^* = 655\,000$ and the total profit is $P^* = 14\,470\,151.95$. This is an optimal solution taking into account the necessity of purchasing fertilizers in 50 kg bags. More precisely, it is assumed to be optimal regarding capabilities of Solver (it cannot be excluded with absolute certainty that some slightly better solution may exist but it might be found with some very special starting values of variables).

The difference in profits to compare with the solution with forced usage of all the purchased fertilizers is $14\,470\,151.95 - 14\,465\,042.66 = 5109.29$. It is worth mentioning that the first profit is achieved at the cost 655 000 while the second profit at the cost 695 000. The excess cost – the difference between the maximal “theoretical” profit (possible only if arbitrary amounts of fertilizers were sold) and the maximal profit for purchase of fertilizers in 50 kg bags is $14\,520\,545.30 - 14\,470\,151.95 = 50\,393.35$.

The latter excess cost is an inevitable result of availability of fertilizers in 50 kg “portions” only. It cannot be neglected in making decisions and it cannot be reduced any more unless there is an opportunity of reselling excess amounts of fertilizers.

4. Conclusions

Achievements in the field of mathematical optimization would be of little value if they were not easily available to real world decision makers. The key factor in the real-world application of optimization techniques is computational complexity of problems which seem to be relatively very simple. Obviously, widespread availability of personal computers and mobile devices like smartphones and tablets can make mathematical optimization useful in everyday operations not only to large corporations and public institutions but also to small businesses. However, availability of hardware must be coupled with availability of cheap/free and user-

friendly optimization software. Optimization of the fertilizer usage is not an exception from the abovementioned rule.

Main approaches to the computational aspects of fertilizer optimization for end users developed so far are spreadsheet-based optimization and dedicated optimization software. Spreadsheet-based optimization can provide optimal (or close to optimal) results at reasonable time and, as it has been shown in this paper, is capable of handling more complex problems than those it has handled so far. However, it has essential disadvantages, too. For technical reasons it requires a personal computer and commercial software (Microsoft Excel). Moreover, built-in optimization capabilities of Excel may not be sufficient if the complexity of the model grows (mainly because of forced limits on the numbers of variables and constraints).

Another approach – dedicated optimization software is probably a better way to match expectations of the potential users. Especially, mobile applications (run on smartphones/tablets) may be of great value because of low cost of the necessary hardware. In both cases a potential issue is a cost of software (appropriately prepared Excel files or standalone optimization applications) since its assumed users can experience essential budget limitations. The case of the OFRA project shows that probably the best way is to pay for such software from public funds and make it free to end users. Helping farmers to make additional profits by subsidizing “transfer of knowledge” can potentially be less costly than possible direct subsidies paid to them. Under the above condition, further development of optimization techniques applicable to the fertilizer usage and all the farming activities in general has a chance not only to be a success of researcher but also to play an important role in improving economy and living standards.

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