Application of the Surface Division Method to Segregate Investments in Capital Markets for Shares' Portfolio

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Abstract:

Purpose: *One of the fundamental issues in capital markets is the sustainability of the price trend. There are many methods of identifying a trend. The article will test a characteristic based on The Surface Division Method.*
Design/Methodology/Approach: *The Surface Division Method is a method that allows for the division of time series into categories due to the reinforcement of the trend, random walk or return to the mean. This fact can be used to segregate investments and choose the right strategy.*
Findings: *The Surface Division Method is a promising method of segregating investments. It is easy to interpret and allows to better describe the shaping of time series values.*
Practical Implications: *The presented investment strategy gave significantly better results than the passive strategy.*
Originality/value: *The Surface Division Method is a new method of data analysis. The application for segregation of investments was made here for the first time. The method is worth developing as it presents a different view than the classical methods based on variance.*

Keywords: *SDM method, portfolio, capital market, time series, trend.*

JEL classification: *G11, G17, C15, C22.*

Paper Type: *Research study.*

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1. Introduction

Contemporary economic processes cannot be explained using classical statistical methods or the basic laws of economics. Many variables affect the phenomena such as the occurrence of business cycles, the situation on the labor market, interest rate or price formation on financial markets. Peters (1997) indicates that forecasts from traditional econometric models do not work in practice or only work in the short term. In addition, a slight change in the initial conditions causes the model to stop functioning properly. Another problem is the assumption of the balance of the economic system, incompatible with the surrounding reality, which is evolving, also causing changes in the system itself (Thalassinos and Kiriazidis, 2003). Therefore, the state of imbalance rather than equilibrium will be characteristic of the economic system.

The development of new scientific disciplines, such as chaos theory and fractal analysis, gives broader possibilities to describe the surrounding reality. A number of relatively simple deterministic chaos systems can be used to model economic phenomena: the logistic equation, the Henon attractor, which is a two-dimensional equivalent of the logistic equation (Mosdorf, 1997), or the Lorenz model (Zawadzki, 1996), known as the butterfly effect. The fractal dimension is also a popular tool derived from fractal geometry.

If the system is a non-linear dynamic system, it is characterized by the occurrence of long-term correlations and trends, unexpected behavior under certain conditions and certain periods, and a structure whose parts (both larger and smaller sections) are similar to the whole and have the same statistical characteristics, this structure is called fractal (Siemieniuk, 2001).

The purpose of the article is the application of the fractal dimension estimated by surface division method for identifying a trend. The use of this can be extended for segregation of investments. With the help of this tool, investment portfolios will be created and it will be checked whether such portfolios provide better results than the passive strategy. The data relates to the Warsaw Stock Exchange.

2. Fractal Dimension of Time Series

In the classical theory of financial investments, one of the most popular risk measures is the variance of returns. The risk is considered to be even greater if the volatility of returns is greater. Interesting solutions in risk estimation are provided by fractal geometry. One of the measures derived from fractal geometry is the fractal dimension of time series, which can complement the classic measures of variation. Although in the literature on the subject, the fractal dimension is often treated as a measure of risk in terms of variability (Zeug-Żebro, 2015), its nature is different. Mandelbrot (1982) gives an example of using the fractal dimension to analyze natural phenomena. The fractal dimension allows you to answer the question of how

jagged coastlines (the more, the larger their fractal dimension). For example, the fractal dimension of the Norwegian coastline is 1.52 and the Great Britain coastline is 1.26. This result is in line with the observations made on the basis of the map, because the Norwegian coastline is more jagged than the coastline of Great Britain, so its fractal dimension is larger and closer to 2.

Today, the fractal dimension is used to describe many natural phenomena (Cervantes-De la Torre, González-Trejo, Real-Ramírez and Hoyos-Reyes, 2013), spatial planning (Guoqiang, 2002; Chen, 2013), medical problems (Gómez, Mediavilla , Hornero, Abásolo and Fernández, 2009; Harne, 2014) or economic (Bhatt, Dedania and Shah, 2015; Kapecka, 2013). The methodology of estimating the fractal dimension is also being developed (Sy-Sang and Feng-Yuan, 2009).

Estimation of the fractal dimension for financial time series requires a departure from the classic Euclidean geometry, giving the dimension of the space in which the time series graph is placed. This space is a plane with the Euclidean dimension 2. While considering the trajectory of the time series as broken, we get the Euclidean dimension 1. Meanwhile, the time series graph does not fill the entire plane on which it was placed, so its dimension will be smaller than 2 (Euclidean dimension of the plane) and larger from 1, because it is a Euclidean straight line dimension, and the time series generally have a different shape.

The fractal dimension as a fractional dimension characterizes the shape of the time series chart. It describes how the time series fills its space and is the result of all factors affecting the system from which the time series originates (Peters, 1997). The effect of the impact of various factors can be a picture of time series of economic variables that will classify them into one of three classes differing in the values of the fractal dimension (Halley and Kunin, 1999):

1. Persistent series (black noise) – in which the phenomenon of trend strengthening is present. This means (in contrast to antipersistent series) that if the value of the series has increased (or decreased) compared to the previous value, then its next increase (or decrease) will be more likely. Such series will have a fractal dimension closer to 1.
2. Random walk time series (white noise) – in which the previous change in value has no effect on future changes. Events affecting series values are accidental and uncorrelated, hence such series are unpredictable.
3. Antipersistent series (pink noise) – in which there is a phenomenon of returning the observation values to the average level. This means that if the value of the series deviates up (or down) from the average, then in the next moment it is more likely to deviate in the opposite direction. Such series will have a fractal dimension closer to 2.

The fractal dimension for financial time series assumes values in the range [1; 2]; 1 – when the graph takes the shape of a straight line, 2 – when it fills a certain two-

dimensional area on the plane. In practice, extreme values are not reached.

Since the fractal dimension is to describe how a time series fills an area, or, in other words, how it compresses on a plane, greater density will increase the fractal dimension. This means that frequent changes of time series in different directions increase the fractal dimension and increase the plane's fill. There is a phenomenon of returning to the mean here. Unidirectional series, with a small number of changes, have smaller fractal dimensions, while their shapes are more similar to a straight shape. This series is characterized by the phenomenon of trend maintenance.

Alternative methods for estimating the fractal dimension include the possible variation method VM (Dubuc, Quiniou, Roques-Carmes, Tricot, and Zucker, 1989). Hurst scaled range analysis is also often used (Hurst, 1951; Kale and Butar Butar, 2011).

3. The Surface Division Method

The method of estimating the fractal dimension presented in the study is based on traditional geometric methods (Przekota and Przekota, 2004). The time series graph is covered by rectangles. The estimation of the fractal dimension itself is done by estimating the regression coefficient. In the following sections, the SDM dimension formula (surface division method) is derived and its practical applications are shown.

Let N be the length of a time series divided into $k = 1, 2, ..., N/2$ parts. The surface area occupied by the series can be defined as:

$$P = N \cdot (y_{max} - y_{min}) \tag{1}$$

where: y_{max} and y_{min} are the highest and the lowest values in the series.

After dividing the series into halves, the surface area will be expressed as:
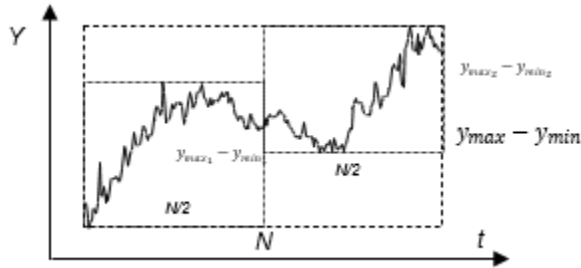
$$p = \frac{N}{2} \cdot \left(y_{max_1} - y_{min_1}\right) + \frac{N}{2} \cdot \left(y_{max_2} - y_{min_2}\right) \tag{2}$$

There is an inequality between p and P:

$$p \leq P \tag{3}$$

This can be seen in figure 1.

Figure 1. *Time series plotted on the plane; surface areas P and p*

Source: *Own study.*

With any primary division into k parts, the surface area occupied by the series is defined as:

$$P_k = \frac{N}{k} \sum_{i=1}^{k} \left(y_{max_i} - y_{min_i} \right) \tag{4}$$

and with a division into $2k$ parts:

$$P_{2k} = \frac{N}{2k} \sum_{i=1}^{2k} \left(y_{max_i} - y_{min_i} \right) \tag{5}$$

There is an inequality between P_k and P_{2k}:

$$P_{2k} \leq P_k \tag{6}$$

Therefore:

$$P_{2k} \leq 2 \cdot \frac{P_k}{2} \tag{7}$$

For example, a series with the length $N = 100$ can be divided into sub-series with the lengths of 100 and 50; 50 and 25; 20 and 10; 10 and 5; 4 and 2.

Then, the following is true for any series:

$$P_{2k} = SDM_k \cdot \frac{P_k}{2} \tag{8}$$

SDM_k is in the range [1; 2] and will be the larger, the more jagged the shape of the time series trajectory, i.e. the more often the trend will change in the opposite series. However, the more the shape of the series approaches a straight line, i.e. the fewer the trend changes in the opposite series occur, the more the SDM_k value will be closer to 1.

If in the coordinate system the values of $P_k/2$ are deposited on the *X*-axis and the values of P_{2k} on the *Y*-axis, the SDM_k value will be the linear regression coefficient of *Y* relative to *X* without a constant, where P_{2k} values play the role of the explained variable, and $P_k/2$ values – the role explanatory variable. Hence:

$$SDM_k = \frac{\sum_k P_{2k} \frac{P_k}{2}}{\sum_k \left(\frac{P_k}{2}\right)^2} \qquad (9)$$

where *k* – the number of divisions made.

The SDM_k value defined in this way can be treated as the fractal dimension of the series. In practical application for data from financial markets, based on the value of the fractal dimension, one can conclude on investment risk.

Two extreme cases of the *SDM* fractal dimension are shown in the fig. 2. The first is a straight line (here it is the function $y = x$). The field after division is always half of the original field, therefore:

$$P_{2k} = 1 \cdot \frac{P_k}{2} \qquad (10)$$

i.e. the fractal dimension here is equal to 1 and is consistent with the Euclidean dimension of the straight line.

The second case is a situation in which the values of the series alternately increase and decrease (here it is 2 for *x* even and 1 for *x* odd). The field after division is always equal to the original field, therefore:

$$P_{2k} = 2 \cdot \frac{P_k}{2} \qquad (11)$$

i.e. the fractal dimension here is equal to 2 and is consistent with the Euclidean dimension of the plane.

In order to distinguish between random walk series from the persistent and antipersistent series, Monte Carlo simulations of the fractal dimension of random walk series were carried out. Table 1 contains the statistics of these simulations together with the results of tests of normality of distribution of the obtained values. A total of 800 simulations were carried out, 100 in each sample. Based on the obtained results of normality tests (Table 1), it was found that the distribution of the fractal dimension is normal, with an average and standard deviation adopted from the average simulated processes. On this basis, tables of significance of the SDM dimension were constructed. These data for selected levels of significance are presented in Table 2. The null hypothesis is verified: the time series generating process is a random walk process.

Figure 2. *Extreme cases of the SDM fractal dimension*
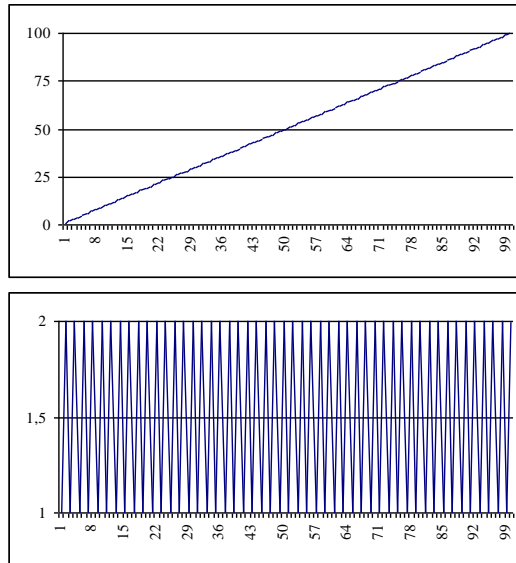
Source: *Own study.*

Table 1. *Statistics of fractal dimension of random walk series simulated by the Monte Carlo method*

Specification	$N = 200$		$N = 500$		$N = 1000$		$N = 1600$	
	Samples							
	1	2	1	2	1	2	1	2
$\bar{x}$	1,3707	1,3731	1,3788	1,3809	1,3843	1,3790	1,3765	1,3703
S	0,1183	0,1140	0,1127	0,1145	0,1100	0,1182	0,1112	0,1017
Minimum	1,1479	1,1485	1,1342	1,1549	1,1580	1,1347	1,1492	1,1573
Maximum	1,6568	1,6594	1,6160	1,6372	1,6125	1,6151	1,6106	1,6041
NT Kołmogorow-Smirnov	$p > 0,20$							
NT Lilliefors	$p > 0,20$							
NT Shapiro-Wilk	$p = 0,1421$	$p = 0,1591$	$p = 0,2305$	$p = 0,3516$	$p = 0,1939$	$p = 0,1716$	$p = 0,2338$	$p = 0,4346$

Note: *$\bar{x}$ – arithmetic mean, S – standard deviation, NT – normality test, p – significance level.*
Source: *Own calculations.*

Depending on the SDM value, three classes of time series can be distinguished:
1. SDM values below the lower limit mean persistent series, i.e. processes with trend strengthening. These are predictable series.
2. SDM values between the lower and upper limits mean series in which the course can be shaped by random walk processes. These are unpredictable.
3. SDM values above the upper limit mean antipersistent series, i.e. processes that characterize the phenomenon of returning to the mean value. They are (like the persistent series) predictable.

Table 2. *Limits for SDM dimension of rejection hypothesis of random walk time series*

N a – lower limit b – upper limit	α = 0,2	α = 0,1	α = 0,05
200 ……………..a	1,2234	1,1813	1,1447
b	1,5204	1,5626	1,5991
500 …………….. a	1,2347	1,1935	1,1578
b	1,5250	1,5662	1,6019
1000 ……............ a	1,2357	1,1942	1,1584
b	1,5277	1,5691	1,6050
1600 ……............ a	1,2371	1,1985	1,1650
b	1,5097	1,5484	1,5819

Note: *α – significance level.*
Source: *Own calculations.*

4.   The Course and Results of the Research

In order to determine the suitability of the Surface Division Method for segregating investments for the share portfolio, simulation studies were carried out using 21 real share portfolios. Each of the portfolios contained 10 randomly selected shares.

The study was conducted in the following order:

1. Selection of the period of time on which the rate of return on investment will be determined. The starting point for this episode was chosen randomly, each investment lasted 2 weeks. Thus, the end point of this episode was determined after 2 weeks.

Table 3 presents the dates of selected time periods and changes in the value of the WIG index (the most general index of the Warsaw Stock Exchange) on these items. These are different two-week periods of 2019. During these periods, the value of the index WIG changed quite significantly, from -5.31% (item no. 7) to 3.99% (item no. 21). Changes in the index WIG can be a good basis for comparing investments in a selected portfolio of shares.

Table 3. *Rates of return of WIG index on selected time periods*

Item no.	Start date	WIG in start date	End date	WIG in end date	Rate of return
1	2019-01-07	58 270,90	2019-01-18	60 289,51	3,46%
2	2019-01-11	59 519,12	2019-01-24	60 791,02	2,14%
3	2019-01-24	60 891,65	2019-02-06	61 319,01	0,70%
4	2019-02-08	60 299,68	2019-02-21	59 938,07	-0,60%
5	2019-02-13	60 718,11	2019-02-26	60 564,73	-0,25%
6	2019-03-29	59 989,13	2019-04-11	61 168,92	1,97%
7	2019-04-18	61 092,17	2019-05-07	57 845,47	-5,31%

8	2019-05-15	56 576,02	2019-05-28	56 920,14	0,61%
9	2019-05-27	56 813,16	2019-06-07	58 852,53	3,59%
10	2019-06-03	57 829,53	2019-06-14	59 092,01	2,18%
11	2019-06-21	59 898,71	2019-07-04	60 888,98	1,65%
12	2019-06-24	59 689,41	2019-07-05	60 628,11	1,57%
13	2019-07-01	60 938,78	2019-07-12	60 378,00	-0,92%
14	2019-07-17	60 248,12	2019-07-30	59 430,05	-1,36%
15	2019-09-17	58 400,61	2019-09-30	57 320,30	-1,85%
16	2019-09-27	57 493,47	2019-10-10	56 034,97	-2,54%
17	2019-10-11	56 238,65	2019-10-24	58 012,97	3,15%
18	2019-10-24	58 054,70	2019-11-08	59 191,71	1,96%
19	2019-11-14	58 799,02	2019-11-25	58 051,64	-1,27%
20	2019-11-14	58 799,02	2019-11-25	58 051,64	-1,27%
21	2019-12-11	55 658,48	2019-12-27	57 877,81	3,99%

Source: Own calculations.

2. Random selection 10 shares in each period of time. Purchase of shares at the beginning of the period and sale of shares at the end of the period.

Table 3. *Rates of return of shares portfolios on selected time periods*

Item no.	Portfolio composition	Rate of return
1	PEO ABE UNI LEN ATT DCR OPL NEU ASB JSW	6,47%
2	JSW UNI SLV TOR NET ABE NEU EAT NTT CER	2,88%
3	KRU IPO NWG GCN ALR PEO TRK ABE STX ING	3,14%
4	PCR ACP URS LEN ING NEU ALR PEO AMC CAR	3,21%
5	STP ZEP LWB WWL IPO AML LPP ACP STX KER	-2,70%
6	JSW OPL AMC NEU NWG ACP ORB BRS RBW ZAP	3,33%
7	ATC GPW ASB TIM ALR AGO DCR AMC MDG KGH	-2,64%
8	MAB ABE GCN DOM ACP ZAP IDA EAT MBK ING	-1,32%
9	MAB ABE RBC IDA UNI SHD ZAP SNK K2I RBW	-2,40%
10	KGH ALR MAB IDA LBT ACP SHD TIM OPL KER	-0,89%
11	WAS ING ERB SKH LPP AMC WLT JSW ATG RBW	3,29%
12	MAB ACP RBC IDA UNI SEN ZAP MBK AAT K2I	2,24%
13	MAB ABE GCN DAT ACP LAB EAT NET ACP PEO	4,14%
14	MBK ATM LBW GTC OPL RDL ZEP ACP TRK AML	-1,04%
15	SNT ZEP LAB WAS IDA ABE STX EAT RBC ACP	-1,63%
16	K2I AAT TSG ALR RFK ZAP ZEP NET ABE CCC	-2,26%
17	ACP KER KTW AML ATT KOM KRU K2I ABE AAT	3,05%
18	GCN RBW ZAP ENG GKI OPL RPC ZEP MBK AMC	3,89%
19	SKH AAT NET DVL RBW ACP JSW AGO RBC ECH	2,08%
20	BCM ABE RBC TIM OPL SHD ZAP K2I UNI BDX	6,06%
21	MBK ACP TOR ENG UNI SNK ZEP GPW ATT WPL	3,88%

Note: 3-letters abbreviations of share names accordance with the Warsaw Stock Exchange.
Source: Own calculations.

The spread of the rates of return obtained from the shares portfolios is smaller than the spread of the WIG indexes and they are higher. The rates of return were in the range from -2.70% (item no. 5) to 6.47% (item no. 1). For 15 items, the rates of return on the shares portfolios are higher than the rates of return on the WIG index, and for 6 items, the rates of return on the shares portfolios are lower than the rates of return on the WIG index. This result was obtained by chance and it cannot mean that a random selection of a portfolio gives better results than the stock index.

3. Extending the series of stock prices with historical data - 1000 observations before the started investment period. An SDM measure was designated for each series.

Figure 3. *Examples of shares prices time series with designated SDM measure*

Source: Own study.

Figure 3 shows an example of stock price time series and the SDM fractal dimension obtained for them. In the first case – DCR share, SDM = 1,14 was obtained. This result proves a strong, statistically significant trend. According to the data from

Table 2: SDM = 1,14 < 1,1584 at α = 0,05. And so is the DCR stock price chart, we have a strong upward trend here. For an investor, this situation means that stock prices are more likely to continue to rise than stock prices will fall.

The second company whose prices are shown in Figure 3 is ASB. Here, the SDM = 1,38 was obtained. This result, according to the data in table 2, classifies the time series as a random walk (1,1584 < 1,38 < 1,6050 at α = 0,05). Also the third company – ALR obtained the SDM result indicating a random walk (1,1584 < 1,52 < 1,6050 at α = 0,05). SDM = 1,38 is closer to 1 than 2, which means that despite the random walk, the time series is trending and so is shown in the chart. However, SDM = 1,52 is in the middle between 1 and 2, so there is no long-term trend, but short-term trends are present. However, in either of these two cases, further development is unpredictable.

Such a procedure, defining the fractal dimension of the SDM, was carried out for the historical quotations of shares of each company. There were 210 time series in total (10 in each of the 21 portfolios). For the limited portfolio, 3 stocks with the lowest fractal SDM dimension were selected from among the 10 considered, i.e. those for which the trend is the strongest (Table 4). Unfortunately, stock price time series are mostly subject to the random walk. Therefore, the selected three also included those with fractal dimensions greater than 1,1584 at α = 0,05.

If the investments were carried out in a manner analogous to that in the original portfolios, i.e. purchase of shares at the beginning of the period and sale of shares at the end of the period, the results would not be good – values in the *Rate of return (mean)* column (Table 4). A better result than the WIG index would be obtained for only 10 items, and a better result than the original portfolio only for 8 items. This is because purchase of shares at the beginning of the period and sale of shares at the end of the period does not take into account the decreasing trend, which always gives a negative result with such an operation. Therefore, in the further part of the research, it was determined in what trend is the share prices and invested in accordance with the trend, i.e.:

- for the growing trend – purchase of shares at the beginning of the period and sale of shares at the end of the period;
- for the declining trend – making the short sale, sale of shares at the beginning of the period and purchase of shares at the end of the period.

Such a procedure definitely improved the results - values in the Rate of return (max) column (Table 4). The result better than the WIG index was obtained for 20 items, and worse only for 1 item, moreover, the result better than the original portfolio was obtained for 18 items, and the result was worse only for 3 items. However, it should be emphasized that the correct identification of the trend is the condition for

obtaining better results. The obtained results should be treated as potentially the best obtainable.

Table 4. *Rates of return of shares for segregation portfolios on selected time periods*

Item no.	Segregated portfolio						Rate of return (mean)	Rate of return (max)
	Share	SDM	Share	SDM	Share	SDM		
1	DCR	1,13	JSW	1,30	OPL	1,32	-0,41%	3,73%
2	EAT	1,09	SLV	1,16	JSW	1,29	0,90%	3,46%
3	ING	1,22	KRU	1,33	TRK	1,33	-1,67%	3,53%
4	PCR	1,13	ING	1,25	URS	1,31	8,88%	9,03%
5	ZEP	1,39	LPP	1,40	STX	1,44	-2,54%	4,43%
6	JSW	1,30	ZAP	1,32	ORB	1,33	0,07%	1,66%
7	DCR	1,14	MDG	1,30	ASB	1,38	-5,77%	7,85%
8	IDA	1,19	EAT	1,21	DOM	1,23	-6,47%	8,82%
9	IDA	1,19	ZAP	1,31	ABE	1,38	0,76%	3,77%
10	IDA	1,19	OPL	1,35	SHD	1,38	4,82%	18,95%
11	ATG	1,17	JSW	1,22	ING	1,29	3,80%	3,80%
12	IDA	1,18	SEN	1,27	ZAP	1,32	2,03%	3,09%
13	EAT	1,20	GCN	1,22	ABE	1,40	10,02%	10,02%
14	RDL	1,12	ATM	1,28	TRK	1,33	-2,71%	4,25%
15	IDA	1,18	EAT	1,23	ABE	1,37	-3,31%	10,13%
16	RFK	1,16	ZAP	1,28	ABE	1,38	5,94%	5,94%
17	KOM	1,28	ATT	1,31	KER	1,37	7,01%	7,01%
18	GCN	1,15	ZAP	1,28	AMC	1,28	2,07%	3,47%
19	NET	1,36	DVL	1,36	ECH	1,37	1,43%	2,89%
20	ZAP	1,26	UNI	1,26	ABE	1,38	5,85%	8,61%
21	UNI	1,25	SNK	1,25	TOR	1,28	3,62%	4,21%

Source: *Own calculations.*

5. Conclusions

The fractal dimension is an interesting alternative to classical methods of assessing the processes of shaping time series values. Its advantages include a fairly simple interpretation: the smaller the dimension, the stronger the phenomenon of maintaining the trend, and the larger the dimension, the stronger the phenomenon of returning to the average value. An additional simplification of interpretation is to normalize the fractal dimension from 1 to 2, i.e. between the Euclidean dimension of the straight line and the plane. A certain limitation on the use of the fractal dimension is the need to study relatively long time series that allow for making divisions.

Investors in the stock market expect changes that are favorable for their value portfolios. Taking the appropriate position on the market - the purchase or sale of shares is dictated by the expected future changes. If investors expect stock prices to

rise, they will buy shares. If investors expect stock prices to fall, they will sell shares. However, it is not easy to make an accurate forecast. There are many methods based on technical analysis and fundamental analysis to support an investor's decisions. The fractal dimension of SDM presented in the article can be treated as a decision supporting tool. The research results are quite promising. They show that the correct interpretation of the trend allows for better investment results. It is admittedly obvious. However, the use of SDM allows us to answer the question of whether the current trend is more likely to be maintained, or whether a random walk is more likely.

The SDM measure allows a better understanding of the shaping of time series values. In a fairly simple way, it is possible to classify a series into the appropriate group due to the presence of a significant trend or a random walk. This undoubted advantage may be helpful for investors in the decision-making process. The method is also developmental. In the work, it was applied to the study of price time series, the nature of which is persistent, but it can also be applied to the time series of price increments, the nature of which is antipersitent.

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